

# On steady state displacements in primary cementing of an oil well

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**Abstract.** Uncontrolled flows of reservoir fluids behind the casing are relatively common in primary cementing and can lead to any of: blow out, leakage at surface, destruction of subsurface ecology, potential contamination of freshwater, delayed or prevented abandonment, as well as loss of revenue due to reduced reservoir pressures. One significant potential cause is ineffective mud removal during primary cementing. Ideally, the drilling mud is displaced all around the annulus and the displacement front advances steadily up the well at the pumping velocity. This paper addresses the question of whether or not such steady state displacements can be found, for a given set of process parameters.

**Keywords:** Hele-Shaw cell, displacement flows, variational methods, visco-plastic fluid flow, steady states, convex analysis

## 1. Introduction

Primary cementing is a critically important operation in the construction of any oil well, [1, 2]. In this process, a steel casing is cemented into the borehole by pumping a sequence of fluids, (e.g. wash, spacer, cement slurry), down the inside of the casing, returning upwards in the annulus, see Fig. 1. The purpose of this operation is to provide a continuous impermeable hydraulic seal in the annulus, preventing any uncontrolled flow of reservoir fluids behind the casing. Many serious problems may arise from uncontrolled flows. Gas or oil may flow to surface causing a blowout, with consequent environmental damage and possible loss of life. Reservoir fluids may migrate into a subsurface aquifer causing contamination of drinking water, or affecting near-wellbore ecology. Finally, even when surface casing vent flows are contained within the annulus, the fact of having pressure at surface prevents a well from being permanently abandoned, (i.e. safely), at the end of its lifetime. Instead, these wells become permanently shut-in and remain

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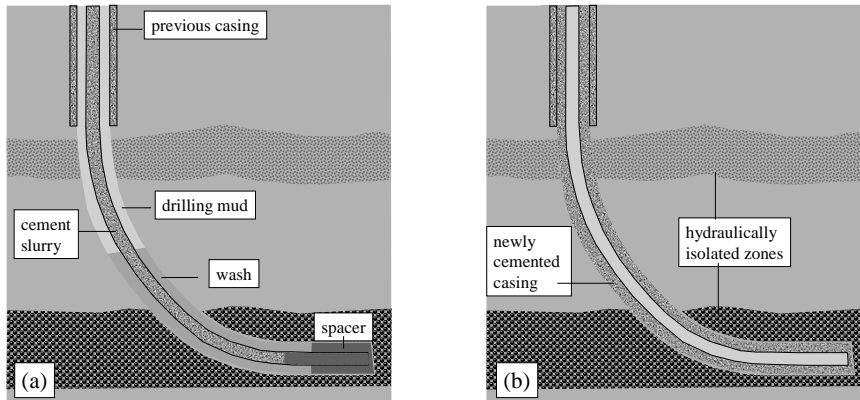


Figure 1. Schematic of the primary cementing process: (a) during and (b) after cementing a new casing.

an environmental risk. From the financial perspective, an hydraulic connection between different fluid bearing zones tends to equilibrate reservoir pressures. Where the higher pressure is the oil-producing zone of the reservoir significant losses in productivity are common.

A widely cited industry figure, e.g. [3], is that 15% of primary cementing jobs carried out in the US fail and that about 1/3 of these failures are due to gas or fluid migration. This problem exists worldwide, (e.g. about 9000 wells are suspended or temporarily abandoned in the U.S. Gulf Coast region). It is also particularly acute in Western Canada, where around 34,000 wells are currently shut-in and suspended, [4]. These are wells that cannot be permanently abandoned due to gas pressures at surface, between the casing and formation, in the cement. Local variations in this problem can be extreme. For example, field survey results are reported in [5], from Tangleflags, Wildmere and Abbey, (3 areas in Eastern Alberta). Over a number of years, 0 – 12% (Tangleflags), 0 – 15% (Wildmere), and 80% (Abbey) of wells are known to be leaking in these regions.

One known cause of surface casing vent flows is that the cement, which is placed in the annulus between the outside of the casing and the inside of the hole, fails to fully displace the drilling mud that initially occupies this space. Most commonly, a channel of mud can be left behind on the narrow (lower) side of the annulus, [6, 7, 8]. As the cement sets, the mud channel becomes dehydrated and porous, allowing reservoir fluids to flow behind the casing along the annulus. This problem, and the consequences described above, provide the main motivation for this paper.

Although various rule-based systems have been developed, e.g. [9, 10], to aid in designing reliable cementing displacements, and these are routinely applied, e.g. [11, 12, 13], they fall far short of providing a complete quantitative description of the process. Such systems are phenomenological in approach and rely usually on an analogy with hydraulic systems. Equally, it is not really feasible to model 3-dimensional displacements along the entire wellbore using computational fluid dynamics software. An intermediate approach to modelling primary cementing is adopted in [14], and it is this approach that we follow here. The model derived in [14] is based on Hele-Shaw approach; its roots may be found in [15], although there the approach is via a porous media analogy. Partial validation and other related experimental work is described in [16, 17].

In this paper we develop further the model derived in [14], with a view to partly addressing the problem of mud-channelling. Ignoring the possibility of leaks to the surrounding formations, whatever fluids are pumped down the inside of the casing end up in the annulus. A sufficient condition, to avoid the problem of mud channelling, is that the displacement proceeds steadily along the annulus in the upwards axial direction, at the mean velocity pumped. This paper addresses the important question of whether or not steady state displacement profiles can exist in the annulus, i.e. travelling wave solutions.

In the model derived in [14] all variables are averaged across the annular gap, eliminating radial velocities. The model consists of a nonlinear elliptic equation for the stream-function and a sequence of advection equations for the fluid concentration. The fluids considered are visco-plastic, meaning here that they will not flow in a narrow channel unless a certain pressure gradient is exceeded, (sufficient for the fluid to yield). The kernel of the stream function equation becomes infinite at points where the fluids do not yield. This is effectively a Hele-Shaw approach. As such, these flows are very similar mathematically to a class of *visco-plastic* porous media flows studied by Entov and co-workers, [18, 19], i.e. those in which there is a limiting pressure gradient.

As a Hele-Shaw displacement problem, there are of course also analogies with classical studies of viscous fingering, [20, 21, 22], and their more recent non-Newtonian studies; see for example [23, 24, 25, 26, 27]. However, the current paper is not focused at instability and viscous fingering, for a number of reasons. First, in these studies the basic displacement flow is straightforward, (e.g. typically planar displacement), whereas here the basic flow is not obvious; indeed finding it is the aim of this paper. Secondly, typical cementing displacements are designed in a sensible way, i.e. heavy viscous fluid displaces upwards the lighter

less viscous fluid. Thus, although instabilities will occur, they may not be *classical* fingering instabilities. Finally, viscous fingering studies are generally local, which have limited relevance over the length-scale of a typical oil well.

A brief outline is as follows. In §2 we present an outline of the model in [14] and extend this to the case where a distinct interface separates the different fluids, §2.1. Section 3 establishes the existence of a solution to the stream function equation, for quite arbitrary and realistic displacements. Steady state solutions are introduced in §4, and we develop analytical solutions for both fully concentric and mildly eccentric annuli, via a perturbation method. Variations in the shapes with the recess parameters are also explored. The paper concludes with a short discussion.

## 2. Model outline

This paper analyses the model for primary cementing that is derived in [14]. Dimensionless spatial coordinates are  $(\phi, \xi) \in (0, 1) \times (0, Z)$ , where  $\phi$  is the azimuthal coordinate;  $\phi = 0$  denotes the *wide side* of the narrow eccentric annular space and  $\phi = 1$  denotes the *narrow side*. Only half of the annulus is considered and it is assumed that the flow is symmetric with the narrow side corresponding to the lower part of the annulus. The  $\xi$  coordinate measures axial depth upwards from bottom-hole ( $\xi = 0$ ) to the top of the zone of interest  $\xi = Z$ , where  $Z \gg 1$ . The radial coordinate, across the annular gap, has been suppressed in the model, by a combination of scaling arguments and averaging. Application of these methods is justified in [14] by the fact that in a typical oil well,

$$[\text{annular gap}] \ll [\text{annular circumference}] \ll [\text{axial length-scale}],$$

is usually satisfied, i.e. it is essentially a lubrication/Hele-Shaw modelling approach in [14]. The annular gap half-width is denoted  $H(\phi, \xi)$ :

$$H(\phi, \xi) = \bar{H}(\xi)(1 + e(\xi) \cos \pi\phi),$$

where  $e(\xi) \in [0, 1)$  is the eccentricity; ( $e(\xi) = 0$ , is concentric and  $e(\xi) = 1$ , implies that the casing contacts the wellbore wall on the narrow side, which we disallow). The mean annular radius at each depth is denoted by  $r_a(\xi)$  and the inclination from vertical is  $\beta(\xi)$ . It is assumed that the annular geometry, (i.e.  $\bar{H}(\xi)$ ,  $e(\xi)$ ,  $r_a(\xi)$  and  $\beta(\xi)$ ), varies slowly in the  $\xi$ -direction, see Fig. 2.

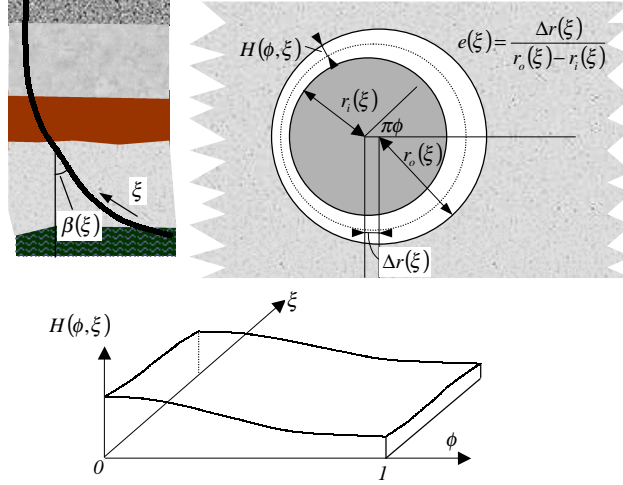


Figure 2. Geometry of the narrow eccentric annulus, mapped to the Hele-Shaw cell geometry.

The flows we consider are therefore two-dimensional and the flow variables have been averaged across the annular gap  $H(\phi, \xi)$ . The continuity equation is simply

$$\frac{\partial}{\partial \phi} [H\bar{v}] + \frac{\partial}{\partial \xi} [Hr_a\bar{w}] = 0, \quad (1)$$

where  $\bar{v}$  &  $\bar{w}$  are the averaged velocities in azimuthal & axial directions, respectively. This prompts definition of a stream-function  $\Psi$ :

$$\frac{\partial \Psi}{\partial \phi} = Hr_a\bar{w}, \quad \frac{\partial \Psi}{\partial \xi} = -H\bar{v}. \quad (2)$$

As is common in a Hele-Shaw model, the (gap-averaged) velocity field is parallel to the (modified) pressure gradient. We consider only laminar flows, and in this case the relationship between the mean speed and the magnitude of the modified pressure gradient,  $G$ , is found directly by considering a Poiseuille flow through a plane channel of separation  $2H$ . For a Newtonian fluid, this relationship can be expressed as:

$$|\nabla_a \Psi| = \frac{H^3 G}{3\mu}, \quad (3)$$

where  $\mu$  denotes the viscosity and  $\nabla_a$  is defined by:

$$\nabla_a q = \left( \frac{1}{r_a(\xi)} \frac{\partial q}{\partial \phi}, \frac{\partial q}{\partial \xi} \right), \quad \nabla_a \cdot \mathbf{q} = \frac{1}{r_a(\xi)} \frac{\partial q_\phi}{\partial \phi} + \frac{\partial q_\xi}{\partial \xi}.$$

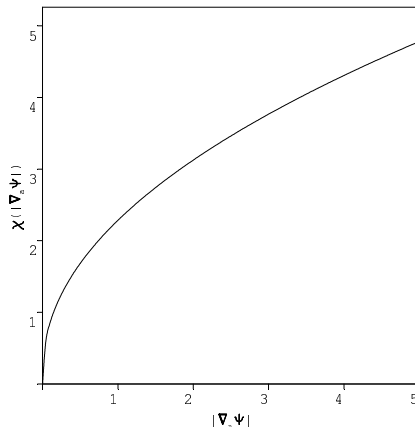


Figure 3. The function  $\chi(|\nabla_a \Psi|; H, \tau_Y, \kappa, m)$ , for parameters  $\tau_Y = \kappa = H = 1.0$ ,  $m = 1/n = 2.0$ .

However, the fluids that we consider are characterised as Herschel-Bulkley fluids, with a yield stress  $\tau_Y$ , consistency  $\kappa$  and power law index  $n$ . If the magnitude of the modified pressure gradient,  $G$ , does not exceed a critical value,  $\tau_Y/H$ , the fluid is unyielded at the walls and does not flow. The excess of the pressure gradient over this critical value is denoted  $\chi$ :

$$\chi = G - \frac{\tau_Y}{H}. \quad (4)$$

The relationship between areal flow rate,  $|\nabla_a \Psi|$ , and  $\chi$  is:

$$|\nabla_a \Psi| = \begin{cases} 0 & \chi \leq 0, \\ \frac{H^{m+2}}{\kappa^m(m+2)} \frac{\chi^{m+1}}{(\chi + \tau_Y/H)^2} \left[ \chi + \frac{(m+2)\tau_Y}{(m+1)H} \right] & \chi > 0, \end{cases} \quad (5)$$

where  $m = 1/n$ , which reduces to (3) when  $n = 1$ ,  $\tau_Y = 0$  and  $\kappa = \mu$ . The function  $\chi$  is defined implicitly as a function of  $|\nabla_a \Psi|$ , by (5). Clearly also we have  $\chi = \chi(|\nabla_a \Psi|; H, \tau_Y, \kappa, m)$ , but we shall be most concerned with the dependence of  $\chi$  on  $|\nabla_a \Psi|$ . In general,  $\chi$  is a positive increasing function of  $|\nabla_a \Psi|$ . An example of this dependence is shown in Fig. 3. Further properties of  $\chi = \chi(|\nabla_a \Psi|)$  are explored in §3.2.1.

In [14] The following coupled system of partial differential equations is derived to model primary cementing displacements.

$$\frac{\partial}{\partial t} [Hr_a \bar{c}_k] + \frac{\partial}{\partial \phi} [H\bar{v} \bar{c}_k] + \frac{\partial}{\partial \xi} [Hr_a \bar{w} \bar{c}_k] = 0, \quad k = 1, 2, \dots, K \quad (6)$$

$$\nabla_a \cdot \mathbf{S} = -f, \quad (7)$$

A sequence of  $K$  fluids is pumped around the wellbore, each with a (radially-averaged) concentration  $\bar{c}_k$ . Diffusive and dispersive effects

are negligible compared to advection. The velocities  $\bar{v}$  and  $\bar{w}$ , in the advection equations (6), are computed from the stream function  $\Psi$ , via (2). The stream function  $\Psi$  is determined from solution of (7) which is coupled with the following visco-plastic constitutive laws (8) & (9):

$$\mathbf{S} = \left[ \frac{r_a \chi (|\nabla_a \Psi|)}{|\nabla_a \Psi|} + \frac{r_a \tau_Y}{H |\nabla_a \Psi|} \right] \nabla_a \Psi \iff |S| > \frac{r_a \tau_Y}{H}, \quad (8)$$

$$|\nabla_a \Psi| = 0 \iff |S| \leq \frac{r_a \tau_Y}{H}. \quad (9)$$

These are been derived by consideration of a Poiseuille flow of an Herschel-Bulkley fluid through a slot of local half-width  $H$ . The term  $f$  in (7) contains the buoyancy terms, given by:

$$f = \nabla_a \cdot \left( \frac{r_a \rho(\bar{c}) \cos \beta}{St^*}, \frac{r_a \rho(\bar{c}) \sin \beta \sin \pi \phi}{St^*} \right) = \nabla_a \cdot \tilde{\mathbf{f}}, \quad (10)$$

where the parameter  $St^*$  is a Stokes number for the displacement flow.

Boundary conditions for (6) are symmetry at the wide and narrow sides and either  $\bar{c}_k = 0$  or  $\bar{c}_k = 1$  at  $\xi = 0$ , according to which fluid is entering the annulus. For (7), which is really an elliptic second order equation, on the wide side of the annulus:

$$\Psi(0, \xi, t) = 0. \quad (11)$$

and on the narrow side, since the flow is incompressible, we have:

$$\Psi(1, \xi, t) = Q(t), \quad (12)$$

where  $Q(t)$  is the dimensionless flow rate. At the ends of the annulus we impose Dirichlet conditions:

$$\Psi(\phi, 0, t) = \Psi_0(\phi, t), \quad (13)$$

$$\Psi(\phi, Z, t) = \Psi_Z(\phi, t), \quad (14)$$

although suitable functions  $\Psi_0(\phi, t)$  &  $\Psi_Z(\phi, t)$  are not always easy to specify.

## 2.1. INTERFACE TRACKING FORMULATION

We note in (6) that each fluid concentration is simply advected through the annulus. Since the inflow concentrations will be specified as either 0 or 1, there is no theoretical possibility for intermediate concentrations to evolve, (although numerically this will happen). It is therefore tempting to track the interface between fluid domains instead of solving (6). Certainly, in considering steady state displacement solutions

analytically, (in §4), it is far simpler to work with an interface position than with a concentration gradient.

It is relatively straightforward to develop an interface-tracking formulation, using similar approximations to those made in [14]. For simplicity, we consider only two fluids and divide the domain  $\Omega = (0, 1) \times (0, Z)$  into two fluid domains:  $\Omega_1$  for the displacing (lower) fluid 1, and  $\Omega_2$  for the displaced (upper) fluid 2. Following the procedure in [14] for this two-fluid problem, (7) becomes:

$$\nabla_a \cdot \mathbf{S}_1 = 0, \quad (\phi, \xi) \in \Omega_1, \quad (15)$$

$$\nabla_a \cdot \mathbf{S}_2 = 0, \quad (\phi, \xi) \in \Omega_2, \quad (16)$$

with  $\mathbf{S}_1$  and  $\mathbf{S}_2$  defined as in (8-9), with properties  $\rho_1, \tau_{1,Y}, \kappa_1, m_1$  in fluid 1 and  $\rho_2, \tau_{2,Y}, \kappa_2, m_2$  in fluid 2. Note from (10) that in each single fluid domain there are no density gradients and thus  $f = 0$ , (we neglect all  $\xi$ -derivatives of the geometrical parameters).

The starting point for modelling the interface evolution is a kinematic equation. After applying scaling arguments and averaging across the gap, the kinematic condition for the interface, denoted by  $\xi = h(\phi, t)$ , becomes:

$$\frac{\partial h}{\partial t} + \frac{\bar{v}}{r_a} \frac{\partial h}{\partial \phi} = \bar{w}, \quad (17)$$

which replaces (6). The leading order continuity conditions at the interface are that the the pressure  $p$  and the stream function  $\Psi$  are continuous across the interface:

$$[p]_1^2 = 0, \quad (18)$$

$$[\Psi]_1^2 = 0, \quad (19)$$

where  $[q]_1^2$  denotes the difference in  $q$  between fluid 2 and fluid 1 across the interface. Since (15) & (16) involve only  $\Psi$ , we would like to replace the pressure in (18) with a closure relation involving only  $\Psi$ . From [14] we can express  $\mathbf{S}_k$  as:

$$\mathbf{S}_k = (S_{k,\phi}, S_{k,\xi}) \equiv \left( -r_a \frac{\partial p}{\partial \xi} - \frac{\rho_k r_a \cos \beta}{St^*} \frac{\partial p}{\partial \phi} - \frac{\rho_k r_a \sin \beta \sin \pi \phi}{St^*} \right). \quad (20)$$

Assuming the smoothness of the interface, the tangential derivative of  $p$  along the interface will also be continuous:

$$[\mathbf{t} \cdot \nabla_a p]_1^2 = \left[ \frac{1}{r_a} \frac{\partial p}{\partial \phi} + \frac{\partial p}{\partial \xi} \frac{\partial h}{\partial \phi} \right]_1^2 = 0 \quad (21)$$



where  $\mathbf{t} = (1, \frac{\partial h}{\partial \phi})$ . Equation (21) allows us to use (8) and (20) to eliminate the pressure, i.e.

$$\left[ \left( \frac{\chi_k + \frac{\tau_{k,Y}}{H}}{|\nabla_a \Psi|} \right) \left( \frac{\partial \Psi}{\partial \xi} - \frac{1}{r_a} \frac{\partial \Psi}{\partial \phi} \frac{\partial h}{\partial \phi} \right) + \left( \frac{\rho_k \sin \beta \sin \pi \phi}{St^*} - \frac{\rho_k \cos \beta}{St^*} \frac{\partial h}{\partial \phi} \right) \right]_1^2 = 0, \quad (22)$$

which expresses (21) in terms of  $\Psi$ . Note that strictly speaking, (22) is valid only if both fluids are yielded at the interface, i.e. if  $|S_k| > r_a \tau_{k,Y}/H$ . Below the yield stress  $S_k$ , and consequently the pressure, are indeterminate and the more general form of (22) is:

$$\left[ \left( \frac{S_{k,\xi}}{r_a} - \frac{S_{k,\phi}}{r_a} \frac{\partial h}{\partial \phi} \right) + \left( \frac{\rho_k \sin \beta \sin \pi \phi}{St^*} - \frac{\rho_k \cos \beta}{St^*} \frac{\partial h}{\partial \phi} \right) \right]_1^2 = 0, \quad (23)$$

which we will use later. Note that (22) describes the jump in the normal derivative of  $\Psi$  at the interface. Finally we note that by differentiating (19) along the interface we have:

$$[\mathbf{t} \cdot \nabla_a \Psi]_1^2 = \left[ \frac{1}{r_a} \frac{\partial \Psi}{\partial \phi} + \frac{\partial \Psi}{\partial \xi} \frac{\partial h}{\partial \phi} \right]_1^2 = 0, \quad (24)$$

which is equivalent to the continuity of the normal component of velocity, which is required for the velocities in (17) to be well-defined.

### 3. Existence of a unique $\Psi$

The aim of the paper is to consider steady state solutions to (6) and (7), i.e. solutions that advance along the annulus at the mean speed of the fluid:  $Q/[r_a \bar{H}]$ . The starting point for answering this question must however be, whether or not there exist solutions to the system (6) & (7), with the specified boundary conditions? Equation (6) is simply an advection equation for the concentration, for which there is little concern. Whether or not solutions to (7) exist is more questionable and we consider this question here.

To motivate this question note that, in either concentration equation or interface-tracking formulation, fluids are simply advected along according to the streamlines  $\Psi$ . The streamfunction is determined from the elliptic problem (7), (or from (15) & (16)), and time only enters in the specification of the flow rate  $Q(t)$  as the narrow side boundary condition. Thus, computing the streamfunction decouples from the time advance of the concentrations and interface. A natural problem therefore, is to assume a realistically regular interface, (or concentration

field), and examine under what conditions a unique stream function solution exists. For the remainder of this section we therefore completely ignore (6), (or the kinematic condition), and consider  $\Psi$  alone.

### 3.1. VARIATIONAL FORMULATION

We observe that the classical formulations that we have derived above are not necessarily well-defined, due to the unyielded regions of the annulus, (where  $|S| \leq r_a \tau_Y / H$ ), being unspecified. Therefore, here we derive a formulation that is more rigorously defined. We proceed purely formally, assuming sufficient regularity of our solution and the test functions that we use. We treat first the fluid mixture formulation and then the interface tracking formulation, showing that both formulations may be described by the same variational inequality.

#### 3.1.1. Fluid mixture formulation

We start with the field equation (7), boundary conditions (11-14) and the constitutive laws (8) & (9). There is no time dependency and thus we consider that a concentration field is defined throughout  $\Omega$ , which is used to define the fluid mixture properties via a set of constitutive laws for each property. Thus, we may consider that each of  $\rho$ ,  $\tau_Y$ ,  $\kappa$ ,  $m$  vary with  $(\phi, \xi)$ . The constitutive laws are unspecified but we do impose the (physically justifiable) restrictions that  $\rho > 0$ ,  $\tau_Y \geq 0$ ,  $\kappa > 0$ ,  $m > 0$ .

We commence by homogenizing the boundary conditions by setting<sup>1</sup>

$$\Psi = \Psi^* + u : \quad u \in C_0^\infty(\Omega). \quad (25)$$

Now let  $v$ ,  $w \in C_0^\infty(\Omega)$ , with  $w = v - u$ . Then from (7) and using the divergence theorem:

$$\int_{\Omega} fw \, d\Omega = \int_{\Omega} \mathbf{S} \cdot \nabla_a w \, d\Omega - \oint_{\partial\Omega} w \mathbf{S} \cdot \mathbf{n} \, ds = \int_{\Omega} \mathbf{S} \cdot \nabla_a w \, d\Omega. \quad (26)$$

Consider the integrand on the right hand side of (26). Firstly, if the fluid is yielded,  $|S| > r_a \tau_Y / H$ :

$$\begin{aligned} \mathbf{S} \cdot \nabla_a w &= \frac{r_a \chi (|\nabla_a \Psi^* + \nabla_a u|)}{|\nabla_a \Psi^* + \nabla_a u|} (\nabla_a \Psi^* + \nabla_a u) \cdot (\nabla_a v - \nabla_a u) \\ &\quad + \frac{\tau_Y r_a}{H} \frac{(\nabla_a \Psi^* + \nabla_a u) \cdot (\nabla_a v - \nabla_a u)}{|\nabla_a \Psi^* + \nabla_a u|} \\ &\leq \frac{r_a \chi (|\nabla_a \Psi^* + \nabla_a u|)}{|\nabla_a \Psi^* + \nabla_a u|} (\nabla_a \Psi^* + \nabla_a u) \cdot (\nabla_a v - \nabla_a u) \\ &\quad + \frac{\tau_Y r_a}{H} (|\nabla_a \Psi^* + \nabla_a v| - |\nabla_a \Psi^* + \nabla_a u|), \end{aligned} \quad (27)$$

using the Cauchy-Schwarz inequality. Secondly, if the fluid is unyielded,  $|\mathbf{S}| \leq r_a \tau_Y / H$ , then  $\nabla_a \Psi = \nabla_a \Psi^* + \nabla_a u = 0$ , and

$$\begin{aligned} \mathbf{S} \cdot \nabla_a w &= \mathbf{S} \cdot [(\nabla_a \Psi^* + \nabla_a v) - (\nabla_a \Psi^* + \nabla_a u)] \leq |\mathbf{S}| |\nabla_a \Psi^* + \nabla_a v| \\ &\leq \frac{\tau_Y r_a}{H} |\nabla_a \Psi^* + \nabla_a v|. \\ &\leq \frac{r_a \chi (|\nabla_a \Psi^* + \nabla_a u|)}{|\nabla_a \Psi^* + \nabla_a u|} (\nabla_a \Psi^* + \nabla_a u) \cdot (\nabla_a v - \nabla_a u) \\ &\quad + \frac{\tau_Y r_a}{H} (|\nabla_a \Psi^* + \nabla_a v| - |\nabla_a \Psi^* + \nabla_a u|), \end{aligned} \quad (28)$$

Finally, using (10) and the divergence theorem, we see that a classical solution of the mixture formulation, will also: satisfy

$$\begin{aligned} \int_{\Omega} \frac{r_a \chi (|\nabla_a \Psi^* + \nabla_a u|)}{|\nabla_a \Psi^* + \nabla_a u|} (\nabla_a \Psi^* + \nabla_a u) \cdot (\nabla_a v - \nabla_a u) \\ + \frac{\tau_Y r_a}{H} (|\nabla_a \Psi^* + \nabla_a v| - |\nabla_a \Psi^* + \nabla_a u|) + \tilde{\mathbf{f}} \cdot \nabla_a (v - u) \, d\Omega \geq 0 \end{aligned} \quad (29)$$

### 3.1.2. Interface tracking formulation

Here we start with the field equations (15) & (16), the constitutive laws (8) & (9), boundary conditions (11-14), and the continuity conditions (19) & (23) at the interface. For simplicity, in our derivation we assume that only 2 fluids are present in the annulus, the generalisation to  $K$  fluids being straightforward. Again we multiply (15) & (16) by a test function  $w \in C_0^\infty(\Omega)$  and integrate over each sub-domain:

$$\int_{\Omega_1} \mathbf{S}_1 \cdot \nabla_a w \, d\Omega_1 + \int_{\Omega_2} \mathbf{S}_2 \cdot \nabla_a w \, d\Omega_2 = \oint_{\partial\Omega_1} w \mathbf{S}_1 \cdot \mathbf{n}_1 \, ds + \oint_{\partial\Omega_2} w \mathbf{S}_2 \cdot \mathbf{n}_2 \, ds \quad (30)$$

corresponding to (26). The boundary integral terms on the right hand side vanish, except along the interface between the two fluids, denoted  $\Gamma$ , where we have

$$\oint_{\partial\Omega_1} w \mathbf{S}_1 \cdot \mathbf{n}_1 \, ds + \oint_{\partial\Omega_2} w \mathbf{S}_2 \cdot \mathbf{n}_2 \, ds = \int_{\Gamma} w (\mathbf{S}_1 - \mathbf{S}_2) \cdot \mathbf{n}_1 \, ds. \quad (31)$$

Substituting for  $\mathbf{S}_k$  and using (23):

$$[\mathbf{S}_1 - \mathbf{S}_2] \cdot \mathbf{n}_1 = -[\tilde{\mathbf{f}}_1 - \tilde{\mathbf{f}}_2] \cdot \mathbf{n}_1 \quad (32)$$

where

$$\tilde{\mathbf{f}}_k = \left( \frac{\rho_k \cos \beta r_a}{St^*}, \frac{r_a \rho_k \sin \beta \sin \pi \phi}{St^*} \right)_j \quad (33)$$

Therefore, using the divergence theorem and noting that  $\nabla_a \cdot \tilde{\mathbf{f}}_k = 0$ , we have:

$$\begin{aligned} \int_{\Gamma} w(\mathbf{S}_1 - \mathbf{S}_2) \cdot \mathbf{n}_1 \, ds &= - \oint_{\partial\Omega_1} w \tilde{\mathbf{f}}_1 \cdot \mathbf{n}_1 \, ds - \oint_{\partial\Omega_2} w \tilde{\mathbf{f}}_2 \cdot \mathbf{n}_2 \, ds \\ &= - \int_{\Omega_1} \tilde{\mathbf{f}}_1 \cdot \nabla_a w \, d\Omega_1 - \int_{\Omega_2} \tilde{\mathbf{f}}_2 \cdot \nabla_a w \, d\Omega_2. \end{aligned} \quad (34)$$

Finally, treating the  $\mathbf{S}_k \cdot \nabla_a w$  terms as with the mixture formulation, we have:

$$\sum_j \int_{\Omega_j} \frac{r_a \chi(|\nabla_a \Psi^* + \nabla_a u|)}{|\nabla_a \Psi^* + \nabla_a u|} (\nabla_a \Psi^* + \nabla_a u) \cdot (\nabla_a v - \nabla_a u) \quad (35)$$

$$+ \frac{\tau_Y r_a}{H} (|\nabla_a \Psi^* + \nabla_a v| - |\nabla_a \Psi^* + \nabla_a u|) + \tilde{\mathbf{f}}_j \cdot \nabla_a (v - u) \, d\Omega_j \geq 0 \quad (36)$$

which is equivalent to the variational formulation (29).

### 3.2. EXISTENCE AND UNIQUENESS

We assume that  $\Psi^*, \Psi \in V$  with  $u \in V_0$ , with  $V_0$  being the closed subspace of  $V$  with  $u = 0$  on  $\partial\Omega$ , (i.e. in the sense of the trace). The spaces  $V$  and  $V_0$  are determined later. Noting the formal equivalence of (29) & (35), we now take

$$\begin{aligned} \int_{\Omega} \frac{r_a \chi(|\nabla_a \Psi^* + \nabla_a u|)}{|\nabla_a \Psi^* + \nabla_a u|} (\nabla_a \Psi^* + \nabla_a u) \cdot (\nabla_a v - \nabla_a u) \\ + \frac{\tau_Y r_a}{H} (|\nabla_a \Psi^* + \nabla_a v| - |\nabla_a \Psi^* + \nabla_a u|) + \tilde{\mathbf{f}} \cdot \nabla_a (v - u) \, d\Omega \geq 0 \\ u \in V_0, \quad \forall v \in V_0, \end{aligned} \quad (37)$$

as the definition of our problem. We assume that each of  $\rho, \tau_Y, \kappa, m \in L^2(\Omega)$ , and hence that  $\tilde{\mathbf{f}} \in [L^2(\Omega)]^2$ . Additionally, we may assume that  $r_a, \beta, e \in L^2(0, Z)$ . It is apparent that a solution to (37) will also minimise the following functional:

$$J(v) = \int_{\Omega} \frac{r_a}{2} \int_0^{|\nabla_a \Psi^* + \nabla_a v|^2} \frac{\chi(s^{1/2})}{s^{1/2}} \, ds + \frac{\tau_Y r_a}{H} |\nabla_a \Psi^* + \nabla_a v| + \tilde{\mathbf{f}} \cdot \nabla_a v \, d\Omega. \quad (38)$$

We now show that the minimization problem

$$J(u) \leq J(v), \quad \forall v \in V_0, \quad u \in V_0; \quad (39)$$

has a unique solution  $u \in V_0$ , which solves the original problem in its variational formulation (29). To do this we use classical results from convex analysis. The application of these results and determination of the space in which the solution lies depends largely on the properties of  $\chi$ . We explore these properties next, and in §3.2.2 give some preliminary results.

### 3.2.1. Properties of $\chi$

Here we are only interested in the properties of  $\chi$  as a function of  $|\nabla_a \Psi|$ . Thus, here we write  $\chi = \chi(|\nabla_a \Psi|)$  and denote by  $\chi'(|\nabla_a \Psi|)$  the partial derivative of  $\chi(|\nabla_a \Psi|)$  with respect to  $|\nabla_a \Psi|$ . Let:

$$A = \frac{H^{m+2}}{\kappa^m(m+2)} > 0; \quad B = \frac{\tau_Y}{H} > 0.$$

The function  $\chi$  is defined implicitly by (5). The following properties are established using elementary methods.

1. Asymptotic behavior as  $|\nabla_a \Psi| \rightarrow 0$ :

$$|\nabla_a \Psi| \sim \frac{A\chi^{m+1}}{B} \left( \frac{m+2}{m+1} \right) \Rightarrow \chi \sim \left( \frac{B(m+1)}{A(m+2)} \right)^{1/(m+1)} |\nabla_a \Psi|^{1/(m+1)} \quad (40)$$

2. Asymptotic behavior as  $|\nabla_a \Psi| \rightarrow \infty$ :

$$|\nabla_a \Psi| \sim A\chi^m \Rightarrow \chi \sim A^{-1/m} |\nabla_a \Psi|^{1/m}. \quad (41)$$

3. Derivatives with respect to  $|\nabla_a \Psi|$  for  $\chi > 0$ :

$$\begin{aligned} \frac{\partial |\nabla_a \Psi|}{\partial \chi} &= \frac{A\chi^m}{(\chi+B)^3} \left( m\chi^2 + 2m \frac{(m+2)}{(m+1)} B\chi + (m+2)B^2 \right) > 0 \\ \chi' &= \frac{\partial \chi}{\partial |\nabla_a \Psi|} = \left( \frac{\partial |\nabla_a \Psi|}{\partial \chi} \right)^{-1} > 0 \end{aligned} \quad (42)$$

i.e.  $\chi$  is strictly increasing.

4. Bounds on  $\chi$ , for all  $|\nabla_a \Psi|$ :

$$\chi \geq \left( \frac{m+1}{A(m+2)} \right)^{1/m} |\nabla_a \Psi|^{1/m} \quad (43)$$

$$\chi \leq \begin{cases} \left( \frac{2B}{A} \right)^{1/(m+1)} |\nabla_a \Psi|^{1/(m+1)} & |\nabla_a \Psi| \leq AB^m/2, \\ \left( \frac{2}{A} \right)^{1/m} |\nabla_a \Psi|^{1/m} & |\nabla_a \Psi| > AB^m/2, \end{cases} \quad (44)$$

## 3.2.2. Preliminary results

PROPOSITION 1. *Let*

$$\Phi(x, y) = \int_0^{x^2+y^2} \frac{\chi(s^{1/2})}{s^{1/2}} ds,$$

with  $\chi$  as above. Then  $\Phi$  is strictly convex in  $\mathbb{R}^2$ .

*Proof.* The Hessian matrix of  $\Phi$  is easily computed and has eigenvalues  $2\frac{\chi(r)}{r}$  and  $2\chi'(r)$ , both of which are strictly positive. Thus  $\Phi(x, y)$  is strictly convex in  $\mathbb{R}^2$ .  $\square$

PROPOSITION 2.  *$J(v)$  as defined in (38) is strictly convex.*

*Proof.* Strict convexity of the first term in  $J(v)$  follows directly from Proposition 1. Convexity of the second term in  $J(v)$  follows from the triangle inequality. The third term in  $J(v)$  is linear in  $v$  and hence trivially convex. Thus,  $J(v)$  is a sum of two convex functions and a strictly convex function and is itself strictly convex in  $\mathbb{R}$ .  $\square$

Now let us show that  $\lim_{\|v\| \rightarrow +\infty} J(v) = +\infty$ . Let

$$K(\mathbf{x}) = \int_{\Omega} \frac{r_a}{2} \int_0^{|\mathbf{x}|^2} \frac{\chi(s^{1/2})}{s^{1/2}} ds + \frac{\tau_Y r_a}{H} |\mathbf{x}| + \tilde{\mathbf{f}} \cdot (\mathbf{x} - \nabla_a \Psi^*) d\Omega, \quad (45)$$

where  $\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  so that  $J(v) = K(\nabla_a \Psi^* + \nabla_a v)$ .

PROPOSITION 3.  *$K(\mathbf{x}) \rightarrow \infty$  as  $\int_{\Omega} |\mathbf{x}|^{1+1/m} d\Omega \rightarrow \infty$ .*

*Proof.* From the lower bounds in §3.2.1, there exists  $\alpha > 0$ :

$$\chi \geq \alpha |\nabla_a \Psi|^{1/m}, \quad (46)$$

Thus,

$$\begin{aligned} K(\mathbf{x}) &\geq \int_{\Omega} \frac{\alpha r_a}{2} \int_0^{|\mathbf{x}|^2} \frac{s^{1/(2m)}}{s^{1/2}} ds - |\tilde{\mathbf{f}}| |\mathbf{x}| + \tilde{\mathbf{f}} \cdot \nabla_a \Psi^* d\Omega \\ &\geq \int_{\Omega} \frac{\alpha r_a}{1+1/m} |\mathbf{x}|^{1+1/m} - |\tilde{\mathbf{f}}| |\mathbf{x}| d\Omega + \delta, \end{aligned} \quad (47)$$

where  $\delta = - \int_{\Omega} \tilde{\mathbf{f}} \cdot \nabla_a \Psi^* d\Omega$ . Using Holder's inequality:

$$\begin{aligned} K(\mathbf{x}) &\geq \tilde{\alpha} \int_{\Omega} |\mathbf{x}|^{1+1/m} d\Omega - c_* \left[ \int_{\Omega} |\mathbf{x}|^{1+1/m} d\Omega \right]^{\frac{1}{1+1/m}} + \delta, \\ &= \tilde{\alpha} \int_{\Omega} |\mathbf{x}|^{1+1/m} d\Omega \left( 1 - \frac{c_*}{\tilde{\alpha}} \left[ \int_{\Omega} |\mathbf{x}|^{1+1/m} d\Omega \right]^{-\frac{1}{1+m}} \right) + \delta, \\ &\rightarrow \infty \text{ as } \int_{\Omega} |\mathbf{x}|^{1+1/m} d\Omega \rightarrow \infty, \end{aligned}$$

where  $\tilde{\alpha} = \min(r_a)\alpha/(1 + 1/m) > 0$ ,  $c_* = \left[ \int_{\Omega} |\tilde{\mathbf{f}}|^{p'} \, d\Omega \right]^{1/p'}$  with  $1/p + 1/p' = 1$  and  $p = 1 + 1/m$ .  $\square$

LEMMA 1.  $J(v) \rightarrow \infty$  as  $\|v\|_{W^{1,1+1/m}} \rightarrow \infty$ .

*Proof.* We note that for  $v \in W_0^{1,1+1/m}(\Omega)$ , a Poincaré-type inequality holds, and the semi-norm

$$\left[ \int_{\Omega} |\nabla v|^{1+1/m} \right]^{\frac{1}{1+1/m}},$$

is an equivalent norm to  $\|v\|_{W^{1,1+1/m}}$ , see for example Theorem 1.7 in chapter 2 of [28]. Secondly, note that as

$$|\nabla_a \Psi^* + \nabla_a v| \geq |\nabla_a v| - |\nabla_a \Psi^*|, \quad (48)$$

we have

$$\|v\|_{W^{1,1+1/m}(\Omega)} \rightarrow \infty \Rightarrow \|\nabla_a \Psi^* + \nabla_a v\|_{L^{1+1/m}(\Omega)} \rightarrow \infty. \quad (49)$$

The result then follows directly from proposition 3.  $\square$

### 3.2.3. Existence and uniqueness

To summarise,  $J(v)$  is continuous, strictly convex and  $J(v) \rightarrow \infty$  as  $\|v\|_{W^{1,1+1/m}(\Omega)} \rightarrow \infty$ . Thus as  $W_0^{1,1+1/m}(\Omega)$  is a closed convex nonempty subset of  $W^{1,1+1/m}(\Omega)$ , from standard optimization theory in Banach spaces it follows that the minimization problem (39) has a unique solution  $u \in W_0^{1,1+1/m}(\Omega)$ , see e.g. [29]. Moreover, on writing  $J(v) = J_0(v) + J_1(v)$  where

$$J_0(v) = \int_{\Omega} \frac{r_a}{2} \int_0^{|\nabla_a \Psi^* + \nabla_a v|^2} \frac{\chi(s^{1/2})}{s^{1/2}} \, ds + \tilde{\mathbf{f}} \cdot \nabla_a v \, d\Omega \quad (50)$$

is strictly convex and Gateaux-differentiable on  $W_0^{1,1+1/m}(\Omega)$  and

$$J_1(v) = \int_{\Omega} \frac{\tau_Y r_a}{H} |\nabla_a \Psi^* + \nabla_a v| \, d\Omega \quad (51)$$

is convex and continuous, we have (see Theorem 2.1 in Chapter V of [30]) that the unique solution of the minimization problem is characterized by

$$(J'_0(u), v - u) + J_1(v) - J_1(u) \geq 0, \quad \forall v \in W_0^{1,1+1/m}(\Omega), \quad u \in W_0^{1,1+1/m}(\Omega) \quad (52)$$

where  $(J'_0(v), w)$  is the Gateaux derivative of  $J_0$ :

$$\begin{aligned} (J'_0(v), w) &= \lim_{t \rightarrow 0} \frac{J_0(v + tw) - J_0(v)}{t} \\ &= \int_{\Omega} \frac{r_a \chi(|\nabla_a \Psi^* + \nabla_a v|)}{|\nabla_a \Psi^* + \nabla_a v|} (\nabla_a \Psi^* + \nabla_a v) \cdot \nabla_a w + \tilde{\mathbf{f}} \cdot \nabla_a w \, d\Omega. \end{aligned}$$

Thus, on substituting into (52) we clearly recover the variational formulation (37). We conclude that the variational formulation has a unique solution in  $W_0^{1,1+1/m}(\Omega)$ . Furthermore, since  $W^{1,1+1/m}(\Omega)$  is an affine space, i.e.

$$W^{1,1+1/m}(\Omega) = \Psi^* + W_0^{1,1+1/m}(\Omega),$$

for any  $\Psi^* \in W^{1,1+1/m}(\Omega)$ , (and we have also shown that such a  $\Psi^*$  can be constructed), it follows that there exists a unique stream function  $\Psi \in W^{1,1+1/m}(\Omega)$ .

**Remark 1:** In proving the above results, for brevity we have been rather negligent in our definition of  $m$ . In the mixture formulation, we must assume (as in [14]), that the rheological parameters of the mixture, (i.e. having intermediate concentrations), are defined by closure laws that preserve positivity of the consistency, power law indices and densities. For example, in [14] a simple linear mixture law was used for simplicity. It's clear from proposition 3 and lemma 1, that we require  $m = \sup_{\Omega} \{m\}$ . Similarly, in the interface-tracking formulation, we take  $m = \max_k \{m_k\}$ . Physically, this corresponds to the fluid that is has the smallest power law index, i.e. is shear thinning to the greatest extent.

**Remark 2:** For displacements involving shear thinning fluids,  $n < 1$ , we have that  $m > 1$  and hence  $W^{1,1+1/m}(\Omega) \not\subset H^1(\Omega)$ , whereas for shear-thickening displacements and those involving Bingham fluids,  $n \geq 1$ , and due to the Sobolev embedding theorems we have  $\Psi \in W^{1,1+1/m}(\Omega) \subset H^1(\Omega)$ .

#### 4. Steady state displacements

The existence and uniqueness results are very general, covering a wide range of geometries as would typically be found in a well. We now address the question of whether a steady state, (travelling wave), solution can be found. For this we shall consider the interface-tracking formulation, both for simplicity and since we will seek an analytical form for the shape of the steady interface. From a physical perspective, such solutions are to be expected in certain situations, e.g. a heavier



more viscous fluid slowly displacing a lighter less viscous fluid upwards in a vertical concentric annulus.

Since the mean axial speed at a depth  $\xi$  in the well is given by  $Q(t)/[\bar{H}(\xi)r_a(\xi)]$ , it is apparent that a steady displacement solution cannot exist if the geometry is changing along the length of the well. Additionally,  $Q(t)$  enters only through the boundary conditions. Therefore, we assume that the well is uniform in the  $\xi$ -direction and that a constant flow rate is pumped. Without loss of generality the scalings in [14] give that:  $\bar{H} = r_a = 1$  and assume  $Q(t) = 1$ . We also assume that  $\beta(\xi)$  and  $e(\xi)$  are constant. This situation is not wholly unrealistic in considering a well, since typically  $Z \gg 1$ , whereas the slowly changing axial geometry changes on an intermediate length-scale,  $L$ :

$$1 \ll L \ll Z,$$

e.g.  $L$  could be related to the length of a stand of casing, the separation between centralisers, or radius of curvature of the drilled hole, all of which a much longer than the circumference, which has been used as the length-scale, (i.e. this ignores  $O(1)$  changes in annular geometry, such as washouts, which in any case violate the model assumptions). Secondly, for laboratory based experimental studies, it is easiest to construct a uniform annulus.

For the interface tracking formulation, we consider only two fluids and introduce a frame of reference  $(z, \phi)$  where  $z = \xi - t$  is moving with the mean speed of the flow, (= unity). The interface separates fluid 1, the lower displacing fluid, from the displaced fluid 2. We consider a long domain  $z \in (-L, L)$ , with the steady interface centred at  $z = 0$ . We denote the interface position in the moving frame by:

$$z = g(\phi, t) = h(\phi, t) - t, \quad (53)$$

and write the stream function  $\Psi$  as

$$\Psi = \Phi + \phi + \frac{e}{\pi} \sin \pi \phi, \quad (54)$$

i.e.  $\Phi$  is the stream function in the moving frame. The kinematic equation (17) becomes

$$(1 + e \cos \pi \phi) \frac{\partial g}{\partial t} - \frac{\partial \Phi}{\partial z} \frac{\partial g}{\partial \phi} = \frac{\partial \Phi}{\partial \phi}, \quad (55)$$

The field equations for  $\Phi$  in the moving frame are (15) & (16), in lower and upper domains, respectively, with constitutive laws (8) & (9). Boundary conditions for  $\Phi$  are:

$$\Phi(0, z) = \Phi(1, z) = 0, \quad (56)$$

*c.f.* conditions (11) & (12), and

$$\frac{\partial \Phi}{\partial z}(\phi, \pm L) = 0, \quad (57)$$

i.e. our section of the annulus is somewhere away from the ends, ( $\xi = 0, Z$ ), and once we move away from the interface a sufficiently large distance,  $L$ , the annulus is filled with a single fluid, flowing parallel to the axis of the annulus. At the interface the following two continuity conditions are satisfied,

$$\begin{aligned} [\Phi]_1^2 &= 0, & (58) \\ \left[ \frac{\rho_k \cos \beta}{St^*} \frac{\partial g}{\partial \phi} - \frac{\rho_k \sin \beta \sin \pi \phi}{St^*} \right]_1^2 &= \left[ \left( \frac{\chi_k + \frac{\tau_{k,Y}}{H}}{|\nabla(\Phi + \phi + \frac{e}{\pi} \sin \pi \phi)|} \right) \times \right. \\ &\quad \left. \left( \frac{\partial \Phi}{\partial z} - \left[ \frac{\partial \Phi}{\partial \phi} + 1 + e \cos \pi \phi \right] \frac{\partial g}{\partial \phi} \right) \right]_1^2 \end{aligned} \quad (59)$$

*c.f.* conditions (19) & (22). Note that for a steady state displacement, both fluids must be yielded at the interface.

In general we will seek steady  $g$  and  $\Phi$ . If  $g$  is steady, then (55) corresponds to saying that the interface is a streamline, (i.e. for  $\Phi$ ). Since the interface must intercept  $\phi = 0$  and  $\phi = 1$ , where (56) is satisfied, it follows that  $\Phi = 0$  on a steady interface. In the following situations we are able to find an analytical solution for both  $g$  and  $\Phi$ .

#### 4.1. CONCENTRIC ANNULI: $e = 0$

This is certainly the simplest case possible and one in which, at least when vertical, our intuition tells us that there must exist a steady state.

Setting  $\Phi = 0$  gives that  $\Psi = \phi$ . Hence  $\nabla \Psi = (1, 0)$ , and since  $\mathbf{S}_k$  is constant in fluid  $k$ , the field equations (15) & (16) are both satisfied. It remains only to satisfy (59), which simplifies to:

$$\left[ \chi_k(1) + \tau_{k,Y} + \frac{\rho_k \cos \beta}{St^*} \right]_1^2 \frac{\partial g}{\partial \phi} = \left[ \frac{\rho_k \sin \beta}{St^*} \right]_1^2 \sin(\pi \phi), \quad (60)$$

which gives the shape of the interface up to an additive constant, (set to zero):

$$g(\phi) = -\frac{1}{\pi} \frac{b \sin \beta}{[\chi_k(1) + \tau_{k,Y}]_1^2 + b \cos \beta} \cos \pi \phi, \quad (61)$$

where  $b$  is the buoyancy parameter:

$$b = \frac{\rho_2 - \rho_1}{St^*}, \quad (62)$$

which will typically be negative since a more dense fluid is usually used to displace a lighter one. Although simplistic, this solution gives a number of insights.

1. The term  $(\chi_k(1) + \tau_{k,Y})$  is the pressure gradient in each fluid, which is in the axial direction only. Thus, the fraction in (61) represents the ratio of differences in the azimuthal and axial modified pressure gradients. If the jump in modified pressure gradients is predominantly axial, the steady state will be *flat*, whereas if the jump is predominantly azimuthal the steady state will elongate along axis of the well. Thus, for a fixed density difference, increasing the frictional pressure drop in the displacing fluid, i.e.  $(\chi_1(1) + \tau_{1,Y})$ , will tend to flatten the steady state profile.
2. Vertical well: Here  $\beta = 0$  giving  $g(\phi) = \text{constant}$ , i.e. the steady-state interface is horizontal as expected.
3. Inclined well: With no rheology difference, we have simply

$$g(\phi) = -\frac{1}{\pi} \tan \beta \cos \pi \phi, \quad (63)$$

which implies that the interface aligns perpendicular to the direction of gravity. Note also that in cementing, the Stokes number  $St^*$  is typically small, so that relatively small density differences can lead to large buoyancy  $b$ . The above shape is that approached in the limit  $b \rightarrow \infty$ , and is independent of the density difference.

4. Horizontal well: Here  $\beta = \pi/2$  and we obtain:

$$g(\phi) = -\frac{1}{\pi} \frac{b}{[\chi_k(1) + \tau_{k,Y}]_1^2} \cos(\pi \phi). \quad (64)$$

In a horizontal well there will naturally be a tendency for the heavier fluid to slump towards the bottom of the annulus. Equation (64) is interesting in that it indicates that these buoyancy effects can be compensated by rheological effects, resulting in a steady state.

Various examples of the variations in (61) are shown in §4.4.

#### 4.2. MILDLY ECCENTRIC ANNULI: $e \ll 1$

In this section, we show that it is also possible to derive an analytical solution in the case that  $e \ll 1$ , by means of a regular perturbation expansion.

In applying this method we linearise with respect to  $e$ , not only about the zero-th order solution, but also about the zero-th order

domain, i.e. steady state shape. Since we wish to find an analytical solution, via separation of variables, we require that the zero-th order domain shape be rectangular, meaning that we must impose the requirement that the zero-th order steady state shape is of  $O(e)$ , (i.e. the fully concentric solution  $g$ , derived in the above section, is small). This requirement is equivalent to assuming that

$$\frac{b \sin \beta}{[\chi_k(1, 1) + \tau_{k,Y}]_1^2 + b \cos \beta} \sin \pi \phi = O(e). \quad (65)$$

Condition (65) is likely to be satisfied either in a nearly-vertical well,  $\beta \ll 1$ , or if the frictional pressure difference is much greater than the buoyancy parameter. Thus, our results will be slightly restrictive. However, in including eccentricity they will still give good insight into the effects of parameter variations on steady state shape.

We seek regular perturbation solutions of form:

$$\begin{aligned} \Phi(z, \phi) &= e\Phi_1(z, \phi) + e^2\Phi_2(z, \phi) + \dots \\ g(\phi) &= eg_1(\phi) + e^2g_2(\phi) + \dots \end{aligned}$$

with

$$\Psi = \Phi + \left(\phi + \frac{e}{\pi} \sin \pi \phi\right),$$

and  $H = 1 + e \cos \pi \phi$ . We substitute these expressions into (15) & (16) and into the rheological laws (8) & (9), expand with respect to small  $e$  and retain only the  $O(e)$  terms, giving:

$$\begin{aligned} (\chi'_k(1, 1) + \chi_{k,H}(1, 1) - \tau_{k,Y})\pi \sin \pi \phi &= (\chi_k(1, 1) + \tau_{k,Y}) \frac{\partial^2 \Phi_1}{\partial z^2} \\ \chi'_k(1, 1) \frac{\partial^2 \Phi_1}{\partial \phi^2} & \quad (66) \end{aligned}$$

in each of the fluid domains  $\Omega_k$ ;  $k = 1, 2$ . Note that we now denote  $\chi_k = \chi_k(|\nabla_a \Psi|, H)$ , since we have linearised both with respect to the solution  $\Phi$  and with respect to the geometry  $H$ . To leading order, our zero-th order solution has  $(|\nabla_a \Psi|, H) = (1, 1)$ . We denote by  $\chi'_k$  the partial derivative of  $\chi_k$  with respect to  $|\nabla_a \Psi|$ , and by  $\chi_{k,H}$ , the partial derivative of  $\chi_k$  with respect to  $H$ .

Equation (66) is a linear elliptic equation in  $\Omega_k$ . Assuming (65), we may linearise the domains, giving to leading order:  $\Omega_1 = (0, 1) \times (-L, 0)$  and  $\Omega_2 = (0, 1) \times (0, L)$ . Boundary conditions at  $\phi = 0, 1$  for the two domains are (56) and at  $z = \pm L$  we have (57). The zero-th order interface is the line  $z = 0$ , along which (58) and (55) imply that

$$\Phi(\phi, 0) = 0, \quad (67)$$

see our earlier discussion. These boundary conditions are sufficient to calculate the solution  $\Phi_1$  to (66) in each domain, (e.g. by homogenizing (66) and using separation of variables). The analytical solutions are:

$$\Phi_1(\phi, z) = \left(1 + \frac{\chi_{1,H}(1, 1) - \tau_{1,Y}}{\chi'_1(1, 1)}\right) \left(-1 + \frac{\cosh \alpha_1(L + z)}{\cosh \alpha_1 L}\right) \frac{\sin \pi \phi}{\pi},$$

$(\phi, z) \in \Omega_1,$  (68)

$$\Phi_1(\phi, z) = \left(1 + \frac{\chi_{2,H}(1, 1) - \tau_{2,Y}}{\chi'_2(1, 1)}\right) \left(-1 + \frac{\cosh \alpha_2(L - z)}{\cosh \alpha_2 L}\right) \frac{\sin \pi \phi}{\pi},$$

$(\phi, z) \in \Omega_2,$  (69)

where

$$\alpha_k^2 = \frac{\pi^2 \chi'_k(1, 1)}{\chi_k(1, 1) + \tau_{k,Y}} > 0. \quad (70)$$

To determine the shape of the interface, we linearise (59), to give:

$$0 = e \left[ (\chi_k(1, 1) + \tau_{k,Y}) \left( \frac{\partial \Phi_1}{\partial z}(\phi, 0) - \frac{\partial g_1}{\partial \phi} \right) - \frac{\rho_k \cos \beta}{St^*} \frac{\partial g_1}{\partial \phi} \right]_1^2$$

$$+ \left[ \frac{\rho_k \sin \beta \sin \pi \phi}{St^*} \right]_1^2 + O(e^2). \quad (71)$$

Note that, because of (65), both terms above have the same order. Rearranging we have:

$$\frac{\partial g_1}{\partial \phi} = \frac{1}{e} \frac{b \sin \beta}{[\chi_k(1, 1) + \tau_{k,Y}]_1^2 + b \cos \beta} \sin \pi \phi$$

$$+ \frac{\left[ (\chi_k(1, 1) + \tau_{k,Y}) \frac{\partial \Phi_1}{\partial z}(\phi, 0) \right]_1^2}{[\chi_k(1, 1) + \tau_{k,Y}]_1^2 + b \cos \beta}. \quad (72)$$

The first term above is simply the concentric annular solution. The second term represents the effect of eccentricity.

To simplify matters, we note that, using (5) and straightforward algebraic methods, that

$$\chi_H(1, 1) - \tau_Y = -\frac{\chi'(1, 1)\chi^m(1, 1)}{\kappa^m}, \quad (73)$$

and that

$$1 - \frac{\chi'(1, 1)\chi^m(1, 1)}{\kappa^m} = -P(\chi(1, 1), \tau_Y, m), \quad (74)$$

where  $P(\chi(1, 1), \tau_Y, m)$  is the strictly positive function given by:

$$P(\chi, \tau_Y, m) = \frac{(m+1)^2 \chi^2 + (m+2)(2m+1)\chi\tau_Y + (m+1)(m+2)\tau_Y^2}{\chi[(m+1)\chi + (m+2)\tau_Y]}. \quad (75)$$

We use (73)-(75), substitute for  $\Phi_1$ , simplify and integrate to give the interface position:

$$g(\phi) \sim -\frac{1}{\pi} \frac{b \sin \beta}{[\chi_k(1, 1) + \tau_{k,Y}]_1^2 + b \cos \beta} \cos \pi \phi \quad (76)$$

$$-\frac{e}{\pi^2} \frac{\sum_{k=1,2} P(\chi_k, \tau_{k,Y}, m_k)(\chi_k + \tau_{k,Y}) \alpha_k \tanh \alpha_k L}{[\chi_k(1, 1) + \tau_{k,Y}]_1^2 + b \cos \beta} \cos \pi \phi,$$

i.e. subject to (65). The simplified solution for  $\Phi$  is

$$\Phi(\phi, z) \sim eP(\chi_1(1, 1), \tau_{1,Y}, m_1) \left( 1 - \frac{\cosh \alpha_1(L+z)}{\cosh \alpha_1 L} \right) \frac{\sin \pi \phi}{\pi} + O(e^2), \quad (\phi, z) \in \Omega_1, \quad (77)$$

$$\Phi(\phi, z) \sim eP(\chi_2(1, 1), \tau_{2,Y}, m_2) \left( 1 - \frac{\cosh \alpha_2(L-z)}{\cosh \alpha_2 L} \right) \frac{\sin \pi \phi}{\pi} + O(e^2), \quad (\phi, z) \in \Omega_2. \quad (78)$$

#### 4.3. REMARKS

The asymptotic solutions for the interface (76) and the stream function in each domain (77) & (78) are interesting primarily because all the problem parameters appear in the solutions, (eccentricity, inclination, density difference, fluid rheologies). Although we require that (65) be satisfied and that  $e \ll 1$ , in order to get quantitatively accurate results, the methodology does not necessarily break down if (65) is not satisfied. Thus, we may consider that the parametric trends remain qualitatively correct at least partly outside of the strict domain of validity or our approximation. To justify this statement, note that if we relax (65) and thus assume

$$\begin{aligned} \Phi(z, \phi) &= e\Phi_1(z, \phi) + e^2\Phi_2(z, \phi) + \dots \\ g(\phi) &= g_0(\phi) + eg_1(\phi) + e^2g_2(\phi) + \dots \end{aligned}$$

this has no effect on (66), i.e. we still solve a linear Poisson equation in  $\Omega_1$  &  $\Omega_2$ . Unfortunately the domains are now:

$$\Omega_1 = (0, 1) \times (-L, g_0(\phi)), \quad \Omega_2 = (0, 1) \times (g_0(\phi), L).$$

This makes analytical solution difficult, but a numerical solution could be fairly easily obtained. The difference now comes in the linearisation

of (59). In place of (71), the zero-th and first order terms are:

$$\frac{\partial g_0}{\partial \phi} = \frac{b \sin \beta}{[\chi_k(1, 1) + \tau_{k,Y}]_1^2 + b \cos \beta} \sin \pi \phi \quad (79)$$

$$\begin{aligned} \frac{\partial g_1}{\partial \phi} = & \frac{\left[ (\chi_k(1, 1) + \tau_{k,Y}) \frac{\partial \Phi_1}{\partial z}(\phi, g_0) - \chi'_k(1, 1) \frac{\partial \Phi_1}{\partial \phi}(\phi, g_0) \right]_1^2}{[\chi_k(1, 1) + \tau_{k,Y}]_1^2 + b \cos \beta} \\ & + \frac{\left[ (\tau_{k,Y} - \chi'_k(1, 1) - \chi_{H,k}(1, 1)) \cos \pi \phi \frac{\partial g_0}{\partial \phi} \right]_1^2}{[\chi_k(1, 1) + \tau_{k,Y}]_1^2 + b \cos \beta} \end{aligned} \quad (80)$$

The zero-th order term is instantly recognisable as (61). The first order term is much more complex than (71). From (80) we can see that higher harmonics will start to appear in our perturbation solution, as is usual. Although numerical solution of (66) and the complexity in (80) prohibit continuing, there is no indication that the perturbation method itself breaks down.

In (76) we can observe the effect of eccentricity on the concentric solution. The numerator of the 2nd term is strictly positive. Intuition tells us that the displacing fluid should be heavier and more *viscous* than the displaced fluid. In this case, the denominator of the second term is negative and we see that the 2nd term in (76) is of form  $e \times C \times \cos \pi \phi$ , for some  $C > 0$ . Thus, in a near vertical well, (so that (65) is valid), for positive density and frictional pressure differences, (as advocated for such field operations, see [10]), the effect of eccentricity is to compensate for the first term in (76) by extending the steady state interface upwards along the wide side.

Much of the algebraic complexity in our solutions is due to the fluids having a yield stress. When there is no yield stress present, explicit formulas can be derived for most of the parameters; (76), (77) & (78) become:

$$g(\phi, z) \sim -\frac{1}{\pi} \frac{b \sin \beta}{[\kappa_k(m_k + 1)^{1/m_k}]_1^2 + b \cos \beta} \cos \pi \phi \quad (81)$$

$$-\frac{e \sum_{k=1,2} \kappa_k(m_k + 1)^{1+1/m_k} m_k^{-1/2} \tanh(\pi m_k^{-1/2} L)}{\pi [\kappa_k(m_k + 1)^{1/m_k}]_1^2 + b \cos \beta} \cos \pi \phi.$$

$$\Phi(\phi, z) \sim e(m_1 + 1) \left( 1 - \frac{\cosh \pi m_1^{-1/2} (L + z)}{\cosh \pi m_1^{-1/2} L} \right) \frac{\sin \pi \phi}{\pi} + O(e^2),$$

$$(\phi, z) \in \Omega_1, \quad (82)$$

$$\Phi(\phi, z) \sim e(m_2 + 1) \left( 1 - \frac{\cosh \pi m_2^{-1/2}(L - z)}{\cosh \pi m_2^{-1/2}L} \right) \frac{\sin \pi \phi}{\pi} + O(e^2),$$

$$(\phi, z) \in \Omega_2. \quad (83)$$

These expressions are valid for power law fluids, (and Newtonian fluids,  $m = 1$ ).

Finally, we may consider what happens in the far-field. For the stream function we may set  $z = \mp L$  in (77) & (78), respectively. At large  $L$  the terms divided by  $\cosh \alpha_k L$  may be neglected, (formally  $L \rightarrow \infty$ ), leaving simply the far field stream functions, which we write as  $\Phi_{\mp\infty}(\phi)$ , given by:

$$\Phi_{-\infty}(\phi) = eP(\chi_1, \tau_{1,Y}, m_1) \frac{\sin \pi \phi}{\pi}, \quad (84)$$

$$\Phi_{\infty}(\phi) = eP(\chi_2, \tau_{2,Y}, m_2) \frac{\sin \pi \phi}{\pi}. \quad (85)$$

The far-field velocities  $\bar{w}_{\mp\infty}(\phi)$  on the wide ( $\phi = 0$ ) and narrow ( $\phi = 1$ ) sides of the annulus (in the original frame of reference) are:

$$\bar{w}_{-\infty}(0) \sim 1 + eP(\chi_1, \tau_{1,Y}, m_1), \quad (86)$$

$$\bar{w}_{-\infty}(1) \sim 1 - eP(\chi_1, \tau_{1,Y}, m_1), \quad (87)$$

$$\bar{w}_{\infty}(0) \sim 1 + eP(\chi_2, \tau_{2,Y}, m_2), \quad (88)$$

$$\bar{w}_{\infty}(1) \sim 1 - eP(\chi_2, \tau_{2,Y}, m_2). \quad (89)$$

Thus, the function  $P(\chi(1, 1), \tau_Y, m)$  measures the maximal velocity differential between the far-field and the steady state. In the case of a power law fluid, we have:

$$P(\chi, \tau_Y, m) = m + 1,$$

and the velocity differential is clearly larger for large  $m$ , i.e. for shear-thinning fluids, as is intuitive. It is interesting to note that in this case the velocity differential depends only on the power law index and not on the consistency of each fluid. For a Newtonian fluid, the ratio of wide to narrow side velocities is simply:  $(1 + 2e)/(1 - 2e)$ .

#### 4.4. ILLUSTRATIVE RESULTS

In Fig. 4 we plot 3 examples of the stream function solution in the moving frame, (77) & (78). Note that our solution is independent of buoyancy and inclination. In Fig. 4a the displacing fluid has significantly higher yield stress, consistency and power law index ( $1/m$ ), than the displaced fluid. The streamlines are more densely packed in



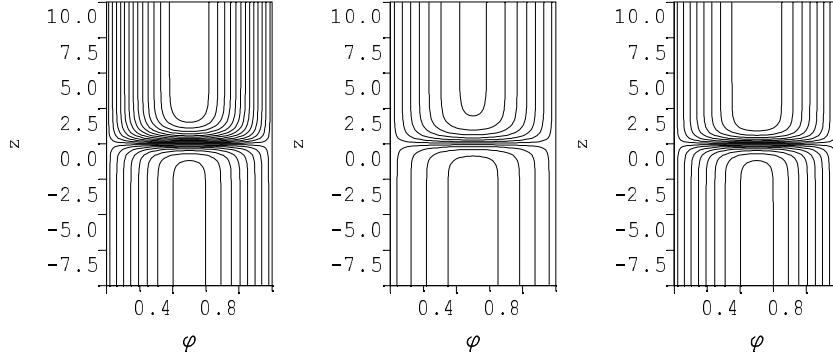


Figure 4. The function  $\Phi$ , for parameters  $L = 10$ ,  $\tau_{1,Y} = \kappa_1 = m_1 = 1.0$ : (left to right) a)  $e = 0.1$ ,  $\tau_{2,Y} = 0.5$ ,  $\kappa_2 = 0.5$ ,  $m_2 = 2.0$ ; b)  $e = 0.05$ ,  $\tau_{2,Y} = 0.5$ ,  $\kappa_2 = 0.5$ ,  $m_2 = 2.0$ ; c)  $e = 0.1$ ,  $\tau_{2,Y} = 0.9$ ,  $\kappa_2 = 0.9$ ,  $m_2 = 1.1$ . Streamlines are spaced at intervals  $\Delta\Phi = 0.01$ .

the *less viscous* upper fluid, indicating that the effects of eccentricity are felt more acutely here. In Fig. 4b the eccentricity has been reduced, giving a consequent (linear in  $e$ ) reduction in the velocity perturbation. Finally, in Fig. 4c we increase the yield stress consistency and power law index of the displaced fluid, to close to that of fluid 1. The streamlines in the two domains are quite similar.

In Figs. 5-10 we explore the variations in (76) with the model parameters. We fix a base case for the parameters:  $\beta = 0.1$ ,  $e = 0.05$ ,  $b = -1$ ,  $\tau_{1,Y} = 1$ ,  $\kappa_1 = 1$ ,  $m_1 = 1$ ,  $\tau_{2,Y} = 0.8$ ,  $\kappa_2 = 0.8$ ,  $m_2 = 1.0$ , and for each of Figs. 5-10 we vary just one parameters to explore the sensitivity. For the rheological parameters, for brevity, we show only the results of varying the properties of fluid 2; qualitatively similar effects are achieved by changing fluid 1 properties in the opposite direction to fluid 2 parameters.

In each of Figs. 5-10 we plot  $g(\phi)$  from (76), and also the functions  $g_{conc}(\phi)$  and  $g_{ecc}(\phi)$ . The function  $g_{conc}(\phi)$  is the concentric annular solution (61), also the first term in (76). The function  $g_{ecc}(\phi)$  is the second term in (76), which contains all the effects of eccentricity. Thus, we compare the effects of each parameter on both the concentric and eccentric terms in (76).

Since our model has assumed (65) to hold, the variations in  $g_{conc}(\phi)$  in Figs. 5-10 are typically smaller than those in  $g_{ecc}(\phi)$ . Often the observed effects are contrary between  $g_{conc}(\phi)$  and  $g_{ecc}(\phi)$ , and due to (65) the eccentric effects dominate. Although eccentricity is certainly an important parameter, one should not conclude that it is always dominant. A better way of interpreting our results is as a competition between  $g_{conc}(\phi)$  and  $g_{ecc}(\phi)$ , i.e. specifically,  $g_{ecc}(\phi)$  is a perturbation

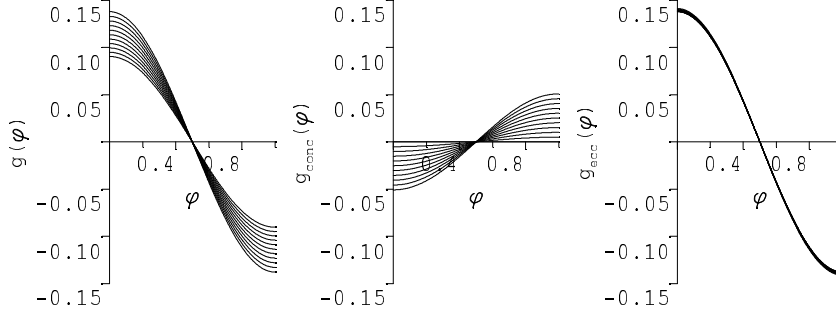


Figure 5. Parametric variations in  $g(\phi)$ ,  $g_{conc}(\phi)$  and  $g_{ecc}(\phi)$  with inclination  $\beta$ :  $\beta = 0, 0.03, 0.06, \dots, 0.3$ ;  $g_{conc}(\phi)$  elongates negatively,  $g_{ecc}(\phi)$  elongates positively. Fixed parameters are:  $e = 0.05$ ,  $b = -1$ ,  $\tau_{1,Y} = 1$ ,  $\kappa_1 = 1$ ,  $m_1 = 1$ ,  $\tau_{2,Y} = 0.8$ ,  $\kappa_2 = 0.8$ ,  $m_2 = 1.0$ .

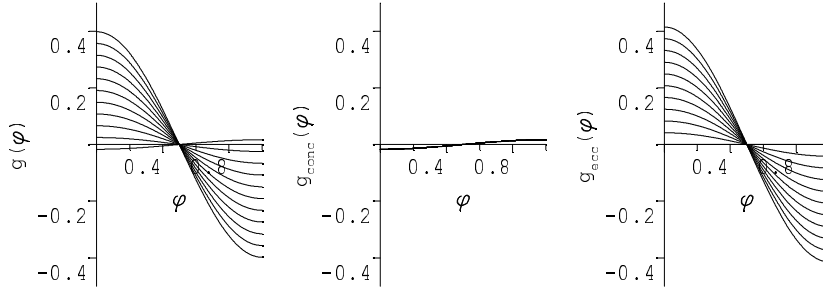


Figure 6. Parametric variations in  $g(\phi)$ ,  $g_{conc}(\phi)$  and  $g_{ecc}(\phi)$  with eccentricity  $e$ :  $e = 0, 0.015, 0.03, \dots, 0.15$ ;  $g_{conc}(\phi)$  is unaffected,  $g_{ecc}(\phi)$  elongates positively. Fixed parameters are  $\beta = 0.1$ ,  $b = -1$ ,  $\tau_{1,Y} = 1$ ,  $\kappa_1 = 1$ ,  $m_1 = 1$ ,  $\tau_{2,Y} = 0.8$ ,  $\kappa_2 = 0.8$ ,  $m_2 = 1.0$ .

of  $g_{conc}(\phi)$ . The relevant feature is usually whether or not increasing a parameter serves to elongate or flatten the steady state profile. These qualitative effects are summarised below.

- $g_{conc}(\phi)$  is elongated negatively, (i.e.,  $g_{conc}(0) - g_{conc}(1) < 0$  and  $|g_{conc}(0) - g_{conc}(1)|$  increases), by the following variations: increasing  $\beta$ ,  $\tau_{2,Y}$ ,  $\kappa_2$ ,  $n_2 = 1/m_2$ ; decreasing  $b$ ,  $\tau_{1,Y}$ ,  $\kappa_1$ ,  $n_1 = 1/m_1$ .
- $g_{ecc}(\phi)$  is elongated positively, (i.e.,  $g_{ecc}(0) - g_{ecc}(1) > 0$  and  $|g_{ecc}(0) - g_{ecc}(1)|$  increases), by the following variations: increasing  $\beta$ ,  $b$ ,  $e$ ,  $\tau_{2,Y}$ ,  $\kappa_2$ ,  $n_2 = 1/m_2$ ; decreasing  $\tau_{1,Y}$ ,  $\kappa_1$ ,  $n_1 = 1/m_1$ .

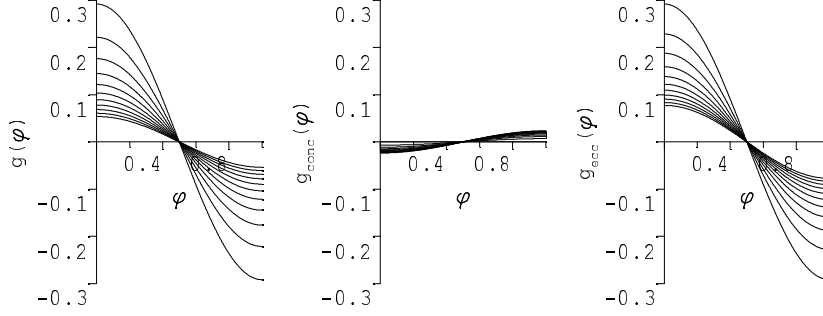


Figure 7. Parametric variations in  $g(\phi)$ ,  $g_{conc}(\phi)$  and  $g_{ecc}(\phi)$  with buoyancy  $b$ :  $b = 0, -0.25, -0.5, \dots, -2.5$ ;  $g_{conc}(\phi)$  elongates negatively,  $g_{ecc}(\phi)$  flattens. Fixed parameters are  $\beta = 0.1$ ,  $e = 0.05$ ,  $\tau_{1,Y} = 1$ ,  $\kappa_1 = 1$ ,  $m_1 = 1$ ,  $\tau_{2,Y} = 0.8$ ,  $\kappa_2 = 0.8$ ,  $m_2 = 1.0$ .

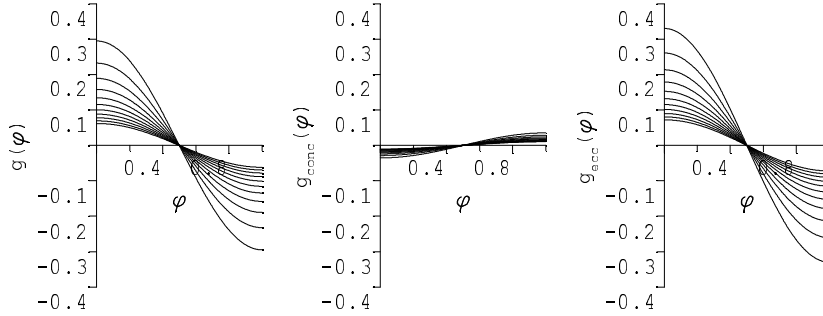


Figure 8. Parametric variations in  $g(\phi)$ ,  $g_{conc}(\phi)$  and  $g_{ecc}(\phi)$  with displaced fluid yield stress  $\tau_{2,Y}$ :  $\tau_{2,Y} = 0, 0.15, 0.3, \dots, 1.5$ ;  $g_{conc}(\phi)$  elongates negatively,  $g_{ecc}(\phi)$  elongates positively. Fixed parameters are  $\beta = 0.1$ ,  $e = 0.05$ ,  $b = -1$ ,  $\tau_{1,Y} = 1$ ,  $\kappa_1 = 1$ ,  $m_1 = 1$ ,  $\kappa_2 = 0.8$ ,  $m_2 = 1.0$ .

In making the above statements, it must be noted that we have explored *small* parametric variations about a set of parameters that might be thought *sensible* to achieve a steady state displacement, i.e. we ensure a hierarchy in both the rheological parameters and densities of the 2 fluids, (heavy & viscous displaces light & less viscous). Inspection of the denominator of the two terms in (76) shows that these solutions can become singular if we consider for example a negative rheology difference versus a positive density difference. Such solutions are not explored here.

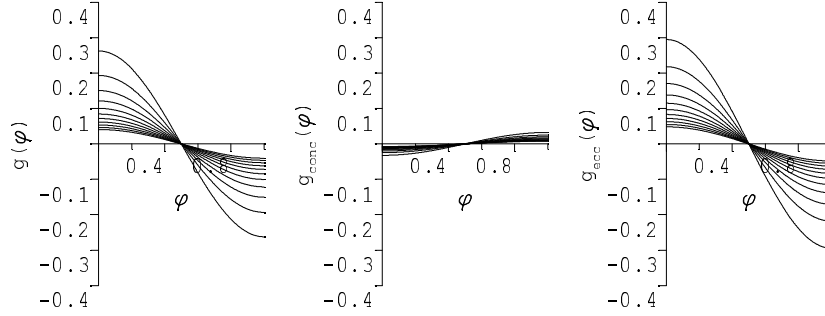


Figure 9. Parametric variations in  $g(\phi)$ ,  $g_{conc}(\phi)$  and  $g_{ecc}(\phi)$  with displaced fluid consistency  $\kappa_2$ :  $\kappa_2 = 0.1, 0.2, 0.3, \dots, 1.1$ ;  $g_{conc}(\phi)$  elongates negatively,  $g_{ecc}(\phi)$  elongates positively. Fixed parameters are  $\beta = 0.1$ ,  $e = 0.05$ ,  $b = -1$ ,  $\tau_{1,Y} = 1$ ,  $\kappa_1 = 1$ ,  $m_1 = 1$ ,  $\tau_{2,Y} = 0.8$ ,  $m_2 = 1.0$ .

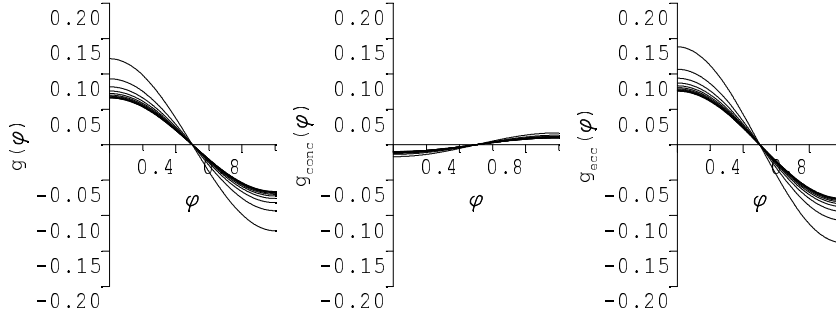


Figure 10. Parametric variations in  $g(\phi)$ ,  $g_{conc}(\phi)$  and  $g_{ecc}(\phi)$  with displaced fluid inverse power law index  $m_2$ :  $m_2 = 1.0, 1.3, 1.6, \dots, 4.0$ ;  $g_{conc}(\phi)$  flattens,  $g_{ecc}(\phi)$  flattens. Fixed parameters are  $\beta = 0.1$ ,  $e = 0.05$ ,  $b = -1$ ,  $\tau_{1,Y} = 1$ ,  $\kappa_1 = 1$ ,  $m_1 = 1$ ,  $\tau_{2,Y} = 0.8$ ,  $\kappa_2 = 0.8$ .

## 5. Discussion and conclusions

This paper has addressed the issue of whether steady state displacements can exist during primary cementing of an oil well. Two questions have been answered affirmatively. Firstly, in §3 we have shown that for more or less any physically sensible distribution of fluid concentrations or fluid interfaces, (i.e. depending on the preferred model formulation), there exists a unique stream function. Obviously, this is a necessary condition to finding any steady solution. Secondly, in §4 we have derived explicit analytic expressions for steady state solutions.

The existence results, although only for a weak solution, are really very comprehensive in terms of likely situations to be encountered in modelling primary cementing. In effect they ensure, from a theoretical standpoint, that the model derived in [14] is mathematically sensible. Physical sensibility of this same model is addressed in [14], where the derivation is careful. Although we have not done so here, it appears possible to derive various results relating to well-posedness of the problem for  $\Psi$ , by considering this variational formulation. A second issue, that we have addressed and which will be presented in a subsequent work, is the question of accurate numerical solution of  $\Psi$ .

Turning to the results in §4, which have already been explored and discussed in some depth, these analytical solutions serve a number of important purposes. First, they confirm unequivocally that steady state displacements can exist in primary cementing. Second, they allow quick comparison with our physical intuition. In particular this is true for the concentric annular solutions. Third, the solutions give a direct quantitative indication of how large competing effects of different parameters are, at least within the range of validity. For example, one might estimate the rheology difference required to counter slumping effects in a horizontal displacement. Fourth, the solutions provide a test case for numerical solutions. We note here that for certain simple cases, (e.g. 2 fluids displacing in a uniform annulus with no intermediate concentrations), a formal equivalence can be established between the interface tracking and fluid mixture formulations.

Despite of the usefulness of §4, the results are in a sense incomplete in that we have not considered the question of stability. It is evident in (61) that if we interchange the fluid properties between fluids 1 and 2, we will leave the shape of steady state unchanged! Certainly, some of these configurations (e.g. a light fluid displacing a heavy fluid, with identical rheology) are simply mechanically unstable, and would not be observed in reality. The question of whether there exist stable steady states is harder to answer, although numerical results shown in [14] do suggest that this is the case. Pragmatically, we may assume that some steady states will be stable, others not.

In the situation of a vertical concentric annulus, one might improve one's understanding of stability. This situation corresponds to a planar displacement along the axis of the annulus. Similarly, an approximately planar displacement can be achieved for certain small  $e$  solutions, where  $g_{conc}(\phi)$  and  $g_{ecc}(\phi)$  approximately balance each other. Local stability analyses of this type of flow have been carried out in [23, 24, 25], in the context of a porous media displacement of non-Newtonian fluids. Recently also stability of planar displacement of a Herschel-Bulkley fluid in a Hele-Shaw cell, (i.e. the Saffman-Taylor problem), has been

partly addressed in [27]; see also the discussion in [26]. Although partly relevant, it must be noted that this type of analysis is local and is focused at instability, meaning the local viscous fingering of the interface. In the context of primary cementing of an oil well, where  $Z \gg 1$ , it is the global behaviour of an instability that is of concern. More specifically, the relevant question is whether or not a displacement results in an unwanted flow structure that extends a significant length along the well. Any “defect” that affects only an axial length of  $O(1)$  is unlikely to compromise the hydraulic isolation of the cement job. In a subsequent paper we will address the prediction of such global defects.

### Acknowledgements:

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### Notes

<sup>1</sup> Note that a suitable  $\Psi^*$  can be defined by taking a linear combination of stream functions at  $\xi = 0$  and  $\xi = Z$ . So in case of two fluids we can take  $\Psi^* = (1 - c)\Psi_0 + c\Psi_Z$ , where  $\Psi_a$  is the stream function at  $\xi = a$ , see (13) & (14) and  $c$  is the concentration with  $c = 0$  for the displacing fluid and  $c = 1$  for the displaced fluid.

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